TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 11, Pages 4455–4472
S 0002-9947(02)03036-2
Article electronically published on July 2, 2002

THE SPACE OF (ψ, γ)–ADDITIVE MAPPINGS ON SEMIGROUPS

VALERIĬ A. FAĬZIEV, THEMISTOCLES M. RASSIAS, AND PRASANNA K. SAHOO

Dedicated to the memory of Maria

Abstract. In this paper, we introduce the concept of (ψ, γ)–pseudoadditive mappings from a semigroup into a Banach space, and we provide a generalized solution of Ulam's problem for approximately additive mappings.

1. Introduction

In 1940, S. M. Ulam [28] posed the following fundamental problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? See S. M. Ulam [28] for a discussion of such problems, as well as D. H. Hyers [15, 16], D. H. Hyers and S. M. Ulam [17, 18], Th. M. Rassias [25, 27], J. Aczél and J. Dhombres [1], I. Fenyö [10], and G. L. Forti [12]. The first affirmative answer was given by D. H. Hyers [15] in 1941.

Theorem 1.1 (Hyers [15])**.** *Let E_1 and E_2 be Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| < \varepsilon \tag{1.1}$$

for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique map $T : E_1 \to E_2$ such that

$$T(x+y) - T(x) - T(y) = 0 \qquad \text{for all} \quad x, y \in E_1 \tag{1.2}$$

and

$$\| f(x) - T(x) \| < \varepsilon \qquad \text{for all} \quad x \in E_1. \tag{1.3}$$

The subject rested there until Th. M. Rassias [25] considered a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$\| f(x+y) - f(x) - f(y) \| \le \varepsilon \left(\|x\|^p + \|y\|^p \right) \qquad \text{for all} \quad x, y \in E_1,$$

where ε and p are constant with $\varepsilon > 0$ and $0 \le p < 1$.

Received by the editors November 20, 2000 and, in revised form, November 5, 2001.
2000 *Mathematics Subject Classification.* Primary 20M15, 20M30, 39B82.
Key words and phrases. Banach spaces, free group, metabelian group, metric group, semigroup, semidirect product of semigroups, stability, wreath product, (ψ, γ)–quasiadditive mapping, (ψ, γ)–pseudoadditive mapping.

By making use of a direct method, Rassias proved that, in this case too, there is an additive function T from E_1 into E_2 given by the formula

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

such that

$$\|T(x) - f(x)\| \leq k\,\varepsilon\,\|x\|^p,$$

where k depends on p as well as ε.

In 1990, during the 27^{th} International Symposium on Functional Equations, Rassias [26] asked whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [13], following the same approach as in [25], gave an affirmative solution to this question for $p > 1$. Several generalizations of these results can be found in [19]–[23] and [25, 26]. In connection with these results the following question arises. Let S be an arbitrary semigroup or group and let a mapping $f : S \to \mathbb{R}$ (the set of reals) be such that the set $\{f(xy) - f(x) - f(y) \,|\, x, y \in S\}$ is bounded. Is it true that there is a mapping $T : S \to \mathbb{R}$ that satisfies

$$T(xy) - T(x) - T(y) = 0 \quad \text{for all } x, y \in S,$$

and that the set $\{T(x) - f(x) \,|\, x \in S\}$ is bounded?

A negative answer was given by G. L. Forti [11] by means of the following example. Let $F(\alpha, \beta)$ be the free group generated by the two elements α, β. Let each word $x \in F(\alpha, \beta)$ be written in reduced form; that is, x does not contain pairs of the forms $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$ and has no exponents different from 1 and -1. Define the function $f : F(\alpha, \beta) \to \mathbb{R}$ as follows. If $r(x)$ is the number of pairs of the form $\alpha\beta$ in x, and $s(x)$ is the number of pairs of the form $\beta^{-1}\alpha^{-1}$ in x, put $f(x) = r(x) - s(x)$. It is easily seen that for all $x, y \in F(\alpha, \beta)$ we have $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$. Now assume that there is a map $T : F(\alpha, \beta) \to \mathbb{R}$ such that the relations (1.2) and (1.3) hold. However, T is completely determined by its values $T(\alpha)$ and $T(\beta)$, while f is identically zero on the subgroups A and B generated by α and β, respectively. For $\alpha \in A$ we have $T(\alpha^n) = nT(\alpha)$ and $f(\alpha^n) = 0$ for $n \in \mathbb{N}$ (the set of natural numbers). Since $T(\alpha^n) - f(\alpha^n) = nT(\alpha)$ for $n \in \mathbb{N}$, it follows that $T(\alpha) = 0$. Similarly we have $T(\beta) = 0$, so that T is identically zero on $F(\alpha, \beta)$. Hence, $f - T = f$ on $F(\alpha, \beta)$, where f is unbounded. This contradiction proves that there is no homomorphism $T : F(\alpha, \beta) \to \mathbb{R}$ such that the relation (1.3) holds.

It turns out that the existence of mappings that are "almost homomorphisms" but are not small perturbations of homomorphisms has an algebraic nature.

Definition 1.2. A *quasicharacter* of a semigroup S is a real-valued function f on S such that the set $\{f(xy) - f(x) - f(y) | x, y \in S\}$ is bounded.

Definition 1.3. By a *pseudocharacter* of a semigroup S (group S) we mean a quasicharacter f that satisfies $f(x^n) = nf(x)$ for all $x \in S$ and all $n \in \mathbb{N}$ (and all $n \in \mathbb{Z}$, if S is a group).

The set of quasicharacters of a semigroup S is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$, and the subspace consisting of real additive characters of the semigroup S will be denoted by $X(S)$. We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$. In connection with the example of Forti [11], note that his function is a quasicharacter of the free group $F(\alpha,\beta)$ but not a pseudocharacter of $F(\alpha,\beta)$. In [5] the set of all pseudocharacters of free groups was described. In [4]–[9] a description of the spaces of pseudocharacters on free groups, semigroups, free products of semigroups, and on some extensions of free groups was given.

For a mapping f of the group G into the semigroup of linear transformations of a vector space, sufficient conditions for the coincidence of the solution of the functional inequality $\|f(xy)-f(x)\cdot f(y)\| < c$ with the solution of the corresponding functional equation $f(xy)-f(x)\cdot f(y)=0$ were studied in [2, 14, 24]. In the papers [14, 24], it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy)-f(x)\cdot f(y)\| \le \delta$ for all $x,y\in G$ with a sufficiently small $\delta>0$, then f is ε-close to a continuous representation g of the same group in the same Banach space (that is, we have $\|f(x)-g(x)\|<\varepsilon$ for all $x\in G$).

In this paper, we introduce the notions of (ψ,γ)–quasiadditive mapping and (ψ,γ)–pseudoadditive mapping. These notions include the notion of pseudocharacter and the notion of ψ–additive mappings. The latter have been introduced in [22] and [23].

2. The space of (ψ,γ)–pseudoadditive mappings

In what follows, by ψ we will mean a function from $\mathbb{R}_+$ (the set of positive reals) to $\mathbb{R}_+$ satisfying the following conditions:

1) ψ is an increasing function,
2) $\psi(t)<t$ for all $t\in\mathbb{R}_+$,
3) $\psi(t_1t_2)\le\psi(t_1)\,\psi(t_2)$,
4) $\psi(t_1+t_2)\le\psi(t_1)+\psi(t_2)$,
5) $\lim\limits_{t\to\infty}\dfrac{\psi(t)}{t}=0$.

Let S be an arbitrary semigroup. By γ we will mean a function from S to $\mathbb{R}_+$ satisfying the inequality

$$\gamma(xy)\le\gamma(x)+\gamma(y).$$

Definition 2.1. Let S be an arbitrary semigroup and E a Banach space. Let the functions ψ and γ satisfy the above conditions. We say that a mapping $f: S\to E$ is a (ψ,γ)–*quasiadditive mapping* if there is a $\theta\in\mathbb{R}_+$ such that

$$\|f(xy)-f(x)-f(y)\|\le\theta\,[\psi(\gamma(x))+\psi(\gamma(y))]\quad\forall x,y\in S. \tag{2.1}$$

It is clear that the set of all (ψ,γ)–additive mappings from S to E is a real linear space relative to the usual operations. Let us denote it by $KAM_{\psi,\gamma}(S;E)$. It is obvious that for any $x\in S$ and for any $m\in\mathbb{N}$ we have

$$\gamma(x^m)\le m\,\gamma(x). \tag{2.2}$$

Proposition 2.2. *Let $f\in KAM_{\psi,\gamma}(S;E)$ and $\theta>0$ be such that*

$$\|f(xy)-f(x)-f(y)\|\le\theta\,[\psi(\gamma(x))+\psi(\gamma(y))]\quad\text{for all}\quad x,y\in S.$$

Then for any $q\in\mathbb{N}$ there is a $c_q\in\mathbb{R}_+$ such that for any $n\in\mathbb{N}$ the following relations are valid:

$$\left\|\frac{1}{q^n}f(x^{q^n})-f(x)\right\|\le c_q\sum_{m=0}^{n-1}\left(\frac{\psi(q)}{q}\right)^m\psi(\gamma(x)) \tag{2.3}$$

and

$$\left\| \frac{1}{q^n} f(x^{q^n}) - f(x) \right\| \leq c_q \, \frac{q}{q - \psi(q)} \, \psi(\gamma(x)). \tag{2.4}$$

Proof. We claim that for any positive integer $q \geq 2$, there is a $c_q > 0$ such that for all $x_1, x_2, \dots, x_q \in S$,

$$\left\| f(x_1 x_2 \cdots x_q) - \sum_{i=1}^{q} f(x_i) \right\| \leq c_q \sum_{i=1}^{q} \psi(\gamma(x_i)). \tag{2.5}$$

The proof follows by induction on q. Obviously (2.5) is true for $q = 2$, and we can assume that $c_2 = \theta$.

Suppose that (2.5) is satisfied for q. We prove it for $q + 1$. We have

$$\begin{aligned}
&\| f(x_1 x_2 \cdots x_q x_{q+1}) - f(x_1 x_2 \cdots x_q) - f(x_{q+1}) \| \\
&\leq c_2 \, [\psi(\gamma(x_1 x_2 \cdots x_q)) + \psi(\gamma(x_{q+1}))] \\
&\leq c_2 \, [\sum_{i=1}^{q} \psi(\gamma(x_i)) + \psi(\gamma(x_{q+1}))] \\
&\leq c_2 \sum_{i=1}^{q+1} \psi(\gamma(x_i)).
\end{aligned}$$

Now from (2.5) we obtain

$$\begin{aligned}
\left\| f(x_1 x_2 \cdots x_q x_{q+1}) - \sum_{i=1}^{q+1} f(x_i) \right\| &\leq c_q \sum_{i=1}^{q} \psi(\gamma(x_i)) + c_2 \sum_{i=1}^{q+1} \psi(\gamma(x_i)) \\
&\leq (c_2 + c_q) \sum_{i=1}^{q+1} \psi(\gamma(x_i)).
\end{aligned}$$

Letting $c_{q+1} = c_2 + c_q$, we have the required result.

Now we prove (2.3). The proof is by induction on n. From (2.5) we obtain

$$\| f(x^q) - q f(x) \| \leq c_q \sum_{i=1}^{q} \psi(\gamma(x)) = c_q \, q \, \psi(\gamma(x)).$$

Therefore,

$$\left\| \frac{1}{q} f(x^q) - f(x) \right\| \leq c_q \psi(\gamma(x)), \tag{2.6}$$

and (2.3) is established for $n = 1$. Next we assume that (2.3) has already been established for n, and prove it for $n + 1$. Substituting x for x^q in (2.3), we get

$$\left\| \frac{1}{q^n} f((x^q)^{q^n}) - f(x^q) \right\| \leq c_q \sum_{m=0}^{n-1} \left(\frac{\psi(q)}{q} \right)^m \psi(\gamma(x^q)).$$

Hence

$$\left\| \frac{1}{q^n} f((x^q)^{q^n}) - f(x^q) \right\| \leq c_q \sum_{m=0}^{n-1} \left(\frac{\psi(q)}{q} \right)^m \psi(q) \, \psi(\gamma(x))$$

and

$$\left\|\tfrac{1}{q^{n+1}} f(x^{q^{n+1}}) - \tfrac{1}{q} f(x^q)\right\| \le c_q \sum_{m=0}^{n-1} \left(\frac{\psi(q)}{q}\right)^{m+1} \psi(\gamma(x))$$
$$= c_q \sum_{m=1}^{n} \left(\frac{\psi(q)}{q}\right)^{m} \psi(\gamma(x)).$$

Thus,

$$\left\|\frac{1}{q^{n+1}} f(x^{q^{n+1}}) - f(x)\right\| \le \left\|\frac{1}{q^{n+1}} f(x^{q^{n+1}}) - \frac{1}{q} f(x^q)\right\| + \left\|\frac{1}{q} f(x^q) - f(x)\right\|$$
$$\le c_q \sum_{m=1}^{n} \left(\frac{\psi(q)}{q}\right)^{m} \psi(\gamma(x)) + c_q\, \psi(\gamma(x))$$
$$\le c_q\, \psi(\gamma(x)) \sum_{m=0}^{n} \left(\frac{\psi(q)}{q}\right)^{m}. \tag{2.7}$$

Hence (2.3) holds for all $n \in \mathbb{N}$. Further, taking into account that $\frac{\psi(q)}{q} < 1$, we have

$$\sum_{m=0}^{\infty} \left(\frac{\psi(q)}{q}\right)^{m} = \frac{q}{q - \psi(q)}, \tag{2.8}$$

and from (2.7) and (2.8), we get (2.4). □

From (2.4) it follows that for any $x \in S$ and any $q \in \mathbb{N}$ the set

$$\left\{\frac{1}{q^n} f(x^{q^n}) \;\middle|\; n \in \mathbb{N}\right\}$$

is bounded.

Lemma 2.3. *Let $f \in KAM_{\psi,\gamma}(S;E)$. For any $x \in S$ and any $q \in \mathbb{N}$, the sequence $\left\{\frac{1}{q^k} f(x^{q^k})\right\}_{k=1}^{\infty}$ has a limit $f_q(x)$. Moreover, $f_q(x^{q^m}) = q^m f_q(x)$ for all $m \in \mathbb{N}$.*

Proof. From (2.4) we have

$$\left\|\frac{1}{q^n} f(x^{q^{n+k}}) - f(x^{q^k})\right\| \le c_q \frac{q}{q - \psi(q)}\, \psi(\gamma(x^{q^k})) \tag{2.9}$$

and

$$\left\|\frac{1}{q^{n+k}} f(x^{q^{n+k}}) - \frac{1}{q^k} f(x^{q^k})\right\| \le c_q \frac{q}{q - \psi(q)} \frac{\psi(q^k)}{q^k}\, \psi(\gamma(x)). \tag{2.10}$$

From (2.10), we conclude that the sequence

$$\left\{\frac{1}{q^k} f(x^{q^k})\right\}_{k=1}^{\infty}$$

is a Cauchy sequence, and by the completeness of E this sequence has a limit. We denote it by $f_q(x)$.

Now we show that for each $m \in \mathbb{N}$, we have $f_q(x^{q^m}) = q^m f_q(x)$. Indeed,

$$f_q(x^{q^m}) = \lim_{k\to\infty} \frac{1}{q^k} f(x^{q^{m+k}}) = q^m \lim_{k\to\infty} \frac{1}{q^{k+m}} f(x^{q^{m+k}}) = q^m f_q(x).$$

□

Lemma 2.4. *$f_q \in KAM_{\psi,\gamma}(S;E)$ for all $q \in \mathbb{N}$.*

Proof. From Proposition 2.2, it follows that there is a $c_q > 0$ such that for any $x \in S$, the inequality

$$\|f_q(x) - f(x)\| \leq c_q \, \frac{q}{q - \psi(q)} \, \psi(\gamma(x))$$

holds. Hence

$$\begin{aligned}
&\|f_q(xy) - f_q(x) - f_q(y)\| \\
&= \|f_q(xy) - f(xy) - f_q(x) + f(x) - f_q(y) - f(y) + f(xy) - f(x) - f(y)\| \\
&\leq \|f_q(xy) - f(xy)\| + \|f_q(x) + f(x)\| \\
&\quad + \|f_q(y) - f(y)\| + \|f(xy) - f(x) - f(y)\| \\
&\leq c_q \frac{q}{q-\psi(q)} \psi(\gamma(xy)) + c_q \frac{q}{q-\psi(q)} \psi(\gamma(x)) \\
&\quad + c_q \frac{q}{q-\psi(q)} \psi(\gamma(y)) + \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\leq c_q \frac{q}{q - \psi(q)} \big[\psi(\gamma(xy)) + \psi(\gamma(x)) + \psi(\gamma(y))\big] + \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\leq c_q \frac{q}{q - \psi(q)} \big[\psi(\gamma(x)) + \psi(\gamma(y)) + \psi(\gamma(x)) + \psi(\gamma(y))\big] \\
&\quad + \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\leq 2c_q \frac{q}{q - \psi(q)} \big[\psi(\gamma(x)) + \psi(\gamma(y))\big] + \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\leq \left(2c_q \frac{q}{q - \psi(q)} + \theta\right) \big[\psi(\gamma(x)) + \psi(\gamma(y))\big].
\end{aligned}$$

□

Lemma 2.5. *Let $f \in KAM_{\psi,\gamma}(S;E)$ and the functions f_q be as in Lemma 2.3. Then*

$$f_2 = f_3 = \cdots = f_m = \cdots .$$

Proof. Let $q > 2$. Let us verify that $f_2 = f_q$. Consider the function φ defined by

$$\varphi(x) = \lim_{m \to \infty} \frac{1}{q^m} f_2(x^{q^m}). \tag{2.11}$$

From Lemma 2.4, we see that $\varphi \in KAM_{\psi,\gamma}(S;E)$. From Lemma 2.3, we have

$$\varphi(x^{q^m}) = q^m \varphi(x)$$

for all $x \in S$ and for all $m \in \mathbb{N}$. Moreover, from the definition of the function φ and formula (2.4) in Proposition 2.2, we obtain

$$\|\varphi(x) - f_2(x)\| \leq c' \, \frac{q}{q - \psi(q)} \, \psi(\gamma(x)) \tag{2.12}$$

for some $c' > 0$ and for all $x \in S$. It is clear from (2.11) that

$$\varphi(x^{2^m}) = 2^m \varphi(x) \tag{2.13}$$

for all $x \in S$ and for all $m \in \mathbb{N}$.

For each $x \in S$ and for $c_q \geq c' \frac{q}{q-\psi(q)}$, from (2.13) and (2.12) it follows that

$$\begin{aligned} 2^n \|\varphi(x) - f_2(x)\| &\leq c \frac{q}{q-\psi(q)} \psi(\gamma(x^{2^n})) \\ &\leq c_q \psi(2^n \gamma(x)) \\ &\leq c_q \psi(2^n) \psi(\gamma(x)). \end{aligned} \tag{2.14}$$

Since

$$\|\varphi(x) - f_2(x)\| \leq c_p \frac{\psi(2^n)}{2^n} \psi(\gamma(x)) \ \to 0 \text{ as } n \to \infty, \tag{2.15}$$

we obtain that $\varphi = f_2$. Similarly, from the relations

$$\varphi(x^{q^m}) = q^m \varphi(x), \quad \forall x \in S, \ \forall m \in \mathbb{N},$$

and

$$\|\varphi(x) - f_q(x)\| \leq c \frac{q}{q-\psi(q)} \psi(\gamma(x)), \quad \forall x \in S,$$

we get $\varphi = f_q$. Thus, we have $f_2 = \varphi = f_q$. □

We denote the function φ introduced in Lemma 2.5 by $\widehat{f}$. In other words, for any $f \in KAM_{\psi,\gamma}(S;E)$ the function $\widehat{f}$ is defined as

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n}). \tag{2.16}$$

Furthermore, for any $f \in KAM_{\psi,\gamma}(S;E)$ the function $\widehat{f}$ satisfies the relation

$$\widehat{f}(x^n) = n\widehat{f}(x), \text{ for } x \in S \text{ and } n \in \mathbb{N}.$$

Indeed,

$$\widehat{f}(x^q) = f_q(x^q) = q f_q(x) = q\widehat{f}(x).$$

Definition 2.6. By a (ψ,γ)–*pseudoadditive mapping* of a semigroup S, we mean a (ψ,γ)–quasiadditive mapping φ such that $\varphi(x^n) = n\varphi(x)$ for all $x \in S$ and for each $n \in \mathbb{N}$.

We denote the space of (ψ,γ)–pseudoadditive mappings of a semigroup S by $PAM_{\psi,\gamma}(S;E)$.

If $E = \mathbb{R}$, then we will call a (ψ,γ)–additive mapping a (ψ,γ)–quasicharacter, and a (ψ,γ)–pseudoadditive mapping a (ψ,γ)–pseudocharacter. We denote the space $KAM_{\psi,\gamma}(S;\mathbb{R})$ by $KX_{\psi,\gamma}(S)$, and $PAM_{\psi,\gamma}(S;\mathbb{R})$ by $PX_{\psi,\gamma}(S)$.

Corollary 2.7. (i) *Let a mapping $f : S \to E$ satisfy the following relation:*

$$\|f(xy) - f(x) - f(y)\| \leq \theta [\psi(\gamma(x)) + \psi(\gamma(y))] \textit{ for all } x, y \in S$$

for some $\theta > 0$. Then $\widehat{f} \in PAM_{\psi,\gamma}(S;E)$, and for $\theta_1 = \theta \frac{2}{2-\psi(2)}$ we have

$$\|\widehat{f}(x) - f(x)\| \leq \theta_1 \psi(\gamma(x)) \textit{ for all } x \in S.$$

(ii) *Let $f \in KX(S)$. Then $\widehat{f} \in PX(S)$. Further, if*

$$\|f(xy) - f(x) - f(y)\| \leq \theta \textit{ for some } \theta > 0 \textit{ and for all } x, y \in S,$$

then

$$\|\widehat{f}(x) - f(x)\| \leq \theta \textit{ for all } x \in S.$$

Clearly, $\widehat{\ }\ :\ KAM_{\psi,\gamma}(S;E) \to PAM_{\psi,\gamma}(S;E)$ is a linear mapping of $KAM_{\psi,\gamma}(S;E)$ onto $PAM_{\psi,\gamma}(S;E)$.

Corollary 2.7 can also be obtained from the results in [3].

Lemma 2.8. *If a semigroup S contains the unit e and $f \in PAM_{\psi,\gamma}(S;E)$, then*

1) $f(e) = 0$, *and*
2) $f(x^{-1}) = -f(x)$ *for any $x \in S$ having an inverse.*

Proof. 1) $nf(e) = f(e^n) = f(e)$ for each $n \in \mathbb{N}$. Hence, $f(e) = 0$.

2) Let $\theta > 0$ be such that

$$\|f(xy) - f(x) - f(y)\| \le \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \qquad \text{for all} \quad x, y \in S. \tag{2.17}$$

Hence, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|f(x^n x^{-n}) - f(x^n) - f(x^{-n})\| \le \theta\,[\psi(\gamma(x^n)) + \psi(\gamma(x^{-n}))],\\
&\|f(e) - f(x^n) - f(x^{-n})\| \le \theta\,[\psi(\gamma(x^n)) + \psi(\gamma(x^{-n}))],\\
&\|f(x^n) + f(x^{-n})\| \le \theta\,[\psi(\gamma(x^n)) + \psi(\gamma(x^{-n}))],\\
&n\|f(x) + f(x^{-1})\| \le \theta\,[\psi(\gamma(x^n)) + \psi(\gamma(x^{-n}))],\\
&n\|f(x) + f(x^{-1})\| \le \theta\,[\psi(n\,\gamma(x)) + \psi(n\,\gamma(x^{-1}))],\\
&n\|f(x) + f(x^{-1})\| \le \theta\,[\psi(n)\,\psi(\gamma(x)) + \psi(n)\,\psi(\gamma(x^{-1}))],\\
&\|f(x) + f(x^{-1})\| \le \theta\,\tfrac{\psi(n)}{n}\,[\psi(\gamma(x)) + \psi(\gamma(x^{-1}))].
\end{aligned}$$

Since $\frac{\psi(n)}{n} \to 0$ as $n \to \infty$, we get $f(x) + f(x^{-1}) = 0$, that is, $f(x^{-1}) = -f(x)$. $\square$

By $B_{\psi,\gamma}(S,E)$ we denote the linear real space of functions on S satisfying the relation

$$\|f(x)\| \le c\,\psi(\gamma(x)) \quad \text{for some } c > 0 \text{ and for all } x \in S.$$

Theorem 2.9. $KAM_{\psi,\gamma}(S;E) = PAM_{\psi,\gamma}(S;E) \oplus B_{\psi,\gamma}(S;E)$.

Proof. It is easy to see that $PAM_{\psi,\gamma}(S;E)$ and $B_{\psi,\gamma}(S;E)$ are subspaces of $KAM_{\psi,\gamma}(S;E)$. Let us show that $PAM_{\psi,\gamma}(S;E) \cap B_{\psi,\gamma}(S;E) = \{0\}$.

Indeed, if $f \in PAM_{\psi,\gamma}(S;E) \cap B_{\psi,\gamma}(S;E)$, then for some $c_f \ge 0$ we have

$$\|f(x)\| \le c_f\,\psi(\gamma(x)), \quad \forall x \in S.$$

Let $q > 1$. Then for each $x \in S$, we get

$$\|f(x^{q^n})\| \le c_f\,\psi(\gamma(x^{q^n})), \quad \forall n \in \mathbb{N}.$$

Therefore,

$$q^n\,\|f(x)\| \le c_f\,\psi(\gamma(x^{q^n})), \quad \forall n \in \mathbb{N},$$

and

$$q^n\,\|f(x)\| \le c_f\,\psi(q^n\,\gamma(x)), \quad \forall n \in \mathbb{N}.$$

From the latter, we get

$$q^n\,\|f(x)\| \le c_f\,\psi(q^n)\,\psi(\gamma(x)), \quad \forall n \in \mathbb{N},$$

and

$$\|f(x)\| \le c_f\,\frac{\psi(q^n)}{q^n}\,\psi(\gamma(x)), \quad \forall n \in \mathbb{N}.$$

Hence $f \equiv 0$. Now let f be an element from $KAM_{\psi,\gamma}(S;E)$; then $\widehat{f} \in PAM_{\psi,\gamma}(S;E)$. From Corollary 2.7, we have $f - \widehat{f} \in B_{\psi,\gamma}(S;E)$. $\square$

Let $\gamma(x) = c$ for some $c > 0$ and for any x from a semigroup S. Then the space $KAM_{\psi,\gamma}(S;E)$ consists of functions $f : S \to E$ such that the set

$$\left\{ f(xy) - f(x) - f(y) \;\middle|\; x, y \in S \right\}$$

is bounded and the set $B_{\psi,\gamma}(S;E)$ consists of a bounded function $\delta : S \to E$.

Now if $E = \mathbb{R}$ we obtain $KAM_{\psi,\gamma}(S;\mathbb{R}) = KX(S)$, $B_{\psi,\gamma}(S;\mathbb{R}) = B(S)$, where $B(S)$ denotes the space of real bounded functions on S.

Theorem 2.10. *Suppose that $f \in PAM_{\psi,\gamma}(S;E)$ and $a, b \in S$. If $ab = ba$, then $f(ab) = f(a) + f(b)$.*

Proof. For any $n \in \mathbb{N}$ we have

$$\begin{aligned}
n \,\|\, f(ab) - f(a) - f(b) \,\| &= \|\, f((ab)^n) - f(a^n) - f(b^n) \,\| \\
&= \|\, f(a^n b^n) - f(a^n) - f(b^n) \,\| \\
&\le \theta \; [\psi(\gamma(a^n)) + \psi(\gamma(b^n))] \\
&\le \theta \, [\psi(\gamma(a^n)) + \psi(\gamma(b^n))] \\
&\le \theta \, [\psi(n\gamma(a)) + \psi(n\gamma(b))] \\
&\le \theta \, \psi(n)[\psi(\gamma(a)) + \psi(\gamma(b))].
\end{aligned}$$

Hence,

$$\|\, f(ab) - f(a) - f(b) \,\| \le \theta \frac{\psi(n)}{n} [\psi(\gamma(a)) + \psi(\gamma(b))].$$

Therefore, $f(ab) - f(a) - f(b) = 0$. ☐

Theorem 2.11. *Suppose that $f \in PAM_{\psi,\gamma}(S;E)$ and $x, y \in S$. Then $f(xy) = f(yx)$.*

Proof. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|f((xy)^{n+1}) - f(x) - f((yx)^n) - f(y)\| \\
&\qquad \le c_3 \, [\psi(\gamma(x)) + \psi(\gamma((yx)^n)) + \psi(\gamma(y))] \\
&\qquad \le c_3 \, [\psi(\gamma(x)) + \psi(n \, \gamma(yx)) + \psi(\gamma(y))] \\
&\qquad \le c_3 \, [\psi(\gamma(x)) + \psi(n) \, \psi(\gamma(yx)) + \psi(\gamma(y))] \\
&\qquad \le c_3 \, [\psi(\gamma(x)) + \psi(n) \, \psi(\gamma(x)) + \psi(n) \, \psi(\gamma(y)) + \psi(\gamma(y))] \\
&\qquad \le c_3 \, (\psi(n) + 1) \, [\psi(\gamma(x)) + \psi(\gamma(y))].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|f((xy)^{n+1}) - f((yx)^{n+1})\| \\
&= \|f((xy)^{n+1}) - f(x) - f(y) - f((yx)^n) + f(x) + f(y) + f((yx)^n) - f((yx)^{n+1})\| \\
&\le \|f((xy)^{n+1}) - f(x) - f(y) - f((yx)^n)\| \\
&\qquad + \|f(x) + f(y) + f((yx)^n) - f((yx)^{n+1})\| \\
&\le c_3 \, (\psi(n) + 1) \, [\psi(\gamma(x)) + \psi(\gamma(y))] + \|f(x) + f(y) - f(yx))\| \\
&\le c_3 \, (\psi(n+1) + 1)[\psi(\gamma(x)) + \psi(\gamma(y))] + c_2 \, [\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\le c_3 \, (\psi(n+1) + 1)[\psi(\gamma(x)) + \psi(\gamma(y))] + c_2 \, (\psi(n+1) + 1) \, [\psi(\gamma(x)) + \psi(\gamma(y))] \\
&\le (c_3 + c_2) \, (\psi(n+1) + 1) \, [\psi(\gamma(x)) + \psi(\gamma(y))]
\end{aligned}$$

and

$$(n+1)\,\|f(xy)-f(yx)\| \le (c_3+c_2)\,(\psi(n+1)+1)\,[\psi(\gamma(x))+\psi(\gamma(y))].$$

Therefore,

$$\|f(xy)-f(yx)\| \le (c_3+c_2)\left(\tfrac{\psi(n+1)}{n+1}+\tfrac{1}{n+1}\right)[\psi(\gamma(x))+\psi(\gamma(y))].$$

Hence, $f(xy)=f(yx)$. □

3. Semidirect product

Suppose that A and B are semigroups with units. Let $G = A\cdot B$ be a semidirect product of A and B, and let A act on B by endomorphisms. Let $a_1,a_2\in A$ and $b_1,b_2\in B$. Then we have $a_1b_1\cdot a_2b_2 = a_1a_2b_1^{a_2}b_2$. Here b^a denotes the image of the element b under the endomorphism a. If $f\in KAM_{\psi,\gamma}(G,E)$, we get

$$\begin{aligned}
&\|f(a_1b_1\cdot a_2b_2)-f(a_1b_1)-f(a_2b_2)\| \le \theta\,[\psi(\gamma(a_1b_1))+\psi(\gamma(a_2b_2))],\\
&\|f(ba)-f(b)-f(a)\| \le \theta\,[\psi(\gamma(b))+\psi(\gamma(a))],\\
&\|f(ab^a)-f(a)-f(b^a)\| \le \theta\,[\psi(\gamma(a))+\psi(\gamma(b^a))].
\end{aligned}$$

Let S be an arbitrary semigroup, $f\in KAM_{\psi,\gamma}(S;E)$, and α an endomorphism of S. Let f^α be a function defined by $f^\alpha(x)=f(x^\alpha)$.

Definition 3.1. Let $f\in KAM_{\psi,\gamma}(S;E)$ and α an endomorphism of a semigroup S. We will say that f is *invariant relative to* α if $f^\alpha = f$. If f is invariant relative to all α from a semigroup A of endomorphisms of S, we will say that f is invariant relative to A. The set of all f in $KAM_{\psi,\gamma}(S;E)$ invariant relative to A will be denoted by $KAM_{\psi,\gamma}(S,A;E)$.

The next corollary follows from Theorem 2.11.

Corollary 3.2. *Let G be an arbitrary group and $f\in PAM_{\psi,\gamma}(G;E)$. Then f is invariant under inner automorphisms of G.*

Proof. Indeed, suppose that α is an inner automorphism of G. Let x be an element of G such that $y^\alpha = x^{-1}yx$ for any $y\in G$. Then by Theorem 2.11, we have $f^\alpha(y)=f(y^\alpha)=f(x^{-1}\cdot yx)=f(yx\cdot x^{-1})=f(y)$. □

It is clear that $KAM_{\psi,\gamma}(S,A;E)$ is a linear space. Its subspace consisting of elements from $PAM_{\psi,\gamma}(S;E)$ will be denoted by $PAM_{\psi,\gamma}(S,A;E)$. Let $f\in PAM_{\psi,\gamma}(B,A;E)$, and let us extend the function f to a function f_1 defined on $G=A\cdot B$ as follows:

$$f_1(ab)=f(b)\quad \forall a\in A,\quad b\in B. \tag{3.1}$$

Lemma 3.3. *Suppose $f\in KAM_{\psi,\gamma}(B,A;E)$, $\gamma(b^a)=\gamma(b)$ and $\gamma(ab)=\gamma(a)+\gamma(b)$ for any $a\in A$ and $b\in B$. Then the function f_1 defined by* (3.1) *is in $KAM_{\psi,\gamma}(G;E)$.*

Proof. Since

$$\begin{aligned}
\|f_1(a_1b_1\cdot a_2b_2)-f_1(a_1b_1)-f_1(a_2b_2)\| &= \|f(b_1^{a_2}b_2)-f(b_1)-f(b_2)\|\\
&= \|f(b_1^{a_2}b_2)-f(b_1^{a_2})-f(b_2)\|\\
&\le \theta\,[\psi(\gamma(b_1^{a_2}))+\psi(\gamma(b_2))]\\
&= \theta\,[\psi(\gamma(b_1))+\psi(\gamma(b_2))]\\
&\le \theta\,[\psi(\gamma(a_1b_1))+\psi(\gamma(a_2b_2))],
\end{aligned}$$

the lemma is proved. □

Lemma 3.4. *Suppose that $f \in PAM_{\psi,\gamma}(B, A; E)$, $\gamma(b^a) = \gamma(b)$, $\gamma(ab) = \gamma(a) + \gamma(b)$, and the function f_1 is defined by (3.1). Then the function*

$$\varphi(x) = \lim_{n\to\infty} \frac{1}{n} f_1(x^n) \quad \forall x \in G = A \cdot B$$

belongs to $PAM_{\psi,\gamma}(G; E)$, and

$$\varphi\big|_B = f, \quad \varphi\big|_A \equiv 0.$$

Proof. It is clear that $\varphi \in PAM_{\psi,\gamma}(G; E)$ and $\varphi\big|_A \equiv 0$. Furthermore, we have

$$\varphi(b) = \lim_{n\to\infty} \frac{1}{n}\, f(b^n) = \lim_{n\to\infty} \frac{1}{n}\, n\, f(b) = f(b).$$

Thus the lemma is proved. □

It is evident that the mapping $f \to \varphi$ is an embedding of $PAM_{\psi,\gamma}(B, A; E)$ into $PAM_{\psi,\gamma}(G; E)$.

Let $f \in PAM_{\psi,\gamma}(A; E)$; then the function $\lambda(ab) = f(a)$ belongs to $PAM_{\psi,\gamma}(G; E)$ and the mapping $f \to \lambda$ is an embedding of $PAM_{\psi,\gamma}(A; E)$ into $PAM_{\psi,\gamma}(G; E)$. Hence, we can assume that the spaces $PAM_{\psi,\gamma}(A; E)$ and $PAM_{\psi,\gamma}(B, A; E)$ are subspaces of $PAM_{\psi,\gamma}(G; E)$.

Theorem 3.5. *Let $\gamma(b^a) = \gamma(b)$, $\gamma(ab) = \gamma(a) + \gamma(b)$, for all $a \in A$ and for all $b \in B$. Then $PAM_{\psi,\gamma}(A \cdot B; E) = PAM_{\psi,\gamma}(A; E) \oplus PAM_{\psi,\gamma}(B, A; E)$.*

Proof. It is clear that $PAM_{\psi,\gamma}(A; E) \cap PAM_{\psi,\gamma}(B, A; E) = \{0\}$. Indeed, if $f \in PAM_{\psi,\gamma}(A; E) \cap PAM_{\psi,\gamma}(B, A; E)$, then $f\big|_A \equiv 0$ and $f\big|_B \equiv 0$. Hence, from the relations

$$\|f(ab) - f(a) - f(b)\| \le \theta\,[\psi(\gamma(a)) + \psi(\gamma(b))],$$
$$\psi(\gamma(a)) \le \psi(\gamma(ab))$$

and

$$\psi(\gamma(b)) \le \psi(\gamma(ab)),$$

we have

$$\|f(ab)\| \le 2\,\theta\,\psi(\gamma(ab)).$$

Therefore, $f \in B_{\psi,\gamma}(G)$ and $f \equiv 0$. Hence, the subspace of $PAM_{\psi,\gamma}(G; E)$ generated by $PAM_{\psi,\gamma}(A; E)$ and $PAM_{\psi,\gamma}(B, A; E)$ is their direct sum; that is,

$$PAM_{\psi,\gamma}(A; E) \oplus PAM_{\psi,\gamma}(B, A; E) \subseteq PAM_{\psi,\gamma}(G; E).$$

Let us verify that

$$PAM_{\psi,\gamma}(A; E) \oplus PAM_{\psi,\gamma}(B, A; E) = PAM_{\psi,\gamma}(G; E).$$

Suppose that $f \in PAM_{\psi,\gamma}(G; E)$ and $\varphi = f\big|_A$; then $\varphi \in PAM_{\psi,\gamma}(A; E)$. Let us extend φ to the function $\widehat{\varphi}$ on the semigroup G as follows:

$$\widehat{\varphi}(ab) = \varphi(a).$$

It is clear that $\widehat{\varphi} \in PAM_{\psi,\gamma}(G; E)$ and

$$g(x) = f(x) - \widehat{\varphi}(x) \in PAM_{\psi,\gamma}(G; E), (f - \widehat{\varphi})\big|_A \equiv 0.$$

Let us verify that $g \in PAM_{\psi,\gamma}(B, A; E)$. Obviously $\widehat{\varphi}\big|_B \equiv 0$. Hence, $g\big|_B \equiv f\big|_B$. Furthermore, for some $\theta > 0$, we have

$$\|g(ba) - g(b) - g(a)\| \leq \theta\, [\psi(\gamma(b)) + \psi(\gamma(a))].$$

Hence,

$$\|g(ab^a) - g(b^a) - g(a)\| \leq \theta\, [\psi(\gamma(b^a)) + \psi(\gamma(a))],$$

and, because $\gamma(b^a) = \gamma(b)$ and $ba = ab^a$, we obtain

$$\|g(b^a) - g(b)\| \leq 2\,\theta\, [\psi(\gamma(b)) + \psi(\gamma(a))].$$

The latter implies that for any $n \in \mathbb{N}$ we have

$$\|g((b^n)^a) - g(b^n)\| \leq 2\,\theta\, [\psi(\gamma(b^n)) + \psi(\gamma(a))].$$

Hence,

$$\begin{aligned} \|g(b^a) - g(b)\| &\leq 2\,\theta\, \tfrac{[\psi(n\,\gamma(b)) + \psi(\gamma(a))]}{n} \\ &\leq 2\,\theta\, \tfrac{\psi(n)}{n}\, \psi(\gamma(b)) + 2\,\theta\, \tfrac{\psi(\gamma(a))}{n}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\psi(\gamma(a))}{n} = 0,$$

we obtain the relation $g(b^a) = g(b)$. Hence, $g \in PAM_{\psi,\gamma}(B, A; E)$. This completes the proof of the theorem. □

4. Stability

Let A be an arbitrary semigroup with unit, and B be a group. For each $b \in B$, denote by $A(b)$ a semigroup that is isomorphic to A under an isomorphism $a \to a(b)$. Denote by $D = A^{(B)} = \prod_{b\in B} A(b)$ the direct product of semigroups $A(b)$. It is clear that if $a(b_1)a(b_2)\cdots a(b_k)$ is an element of D, then for any $b \in B$, the mapping

$$b^* : a(b_1)a(b_2)\cdots a(b_k) \to a(b_1 b)a(b_2 b)\cdots a(b_k b)$$

is an automorphism of D and $b \to b^*$ is an embedding of B into $\operatorname{Aut} D$.

Hence, we can form a semidirect product $G = B \cdot D$. This semigroup is the *wreath product* of the semigroup A and the group B, and will be denoted by $G = A \wr B$. We will identify the semigroup A with $A(1)$, where $1 \in B$. Hence, we can assume that A is a subsemigroup of D.

Let $\gamma_A : A \to \mathbb{R}_+$ and $\gamma_A(xy) \leq \gamma_A(x) + \gamma_A(y)$ for all $x, y \in A$. Let $\gamma_B : B \to \mathbb{R}_+$ be such that $\gamma_B(xy) \leq \gamma_B(x) + \gamma_B(y)$ for all $x, y \in B$. Let us extend γ_A from A to D as follows: $\gamma(a_1(b_1)a_2(b_2)\cdots a_m(b_m)) = \sum_{i=1}^m \gamma_A(a_i)$.

Now let us write

$$\gamma(b \cdot a_1(b_1)a_2(b_2)\cdots a_m(b_m)) = \gamma_B(b) + \gamma(a_1(b_1)a_2(b_2)\cdots a_m(b_m)).$$

Denote by $HOM(A; E)$ the set of all homomorphisms from A into E.

Theorem 4.1. (i) *If the group B is infinite, then*

$$PAM_{\psi,\gamma}(A \wr B) = PAM_{\psi,\gamma}(B) \oplus HOM(A; E).$$

(ii) *If the group B is finite, then*

$$PAM_{\psi,\gamma}(A \wr B; E) = PAM_{\psi,\gamma}(B; E) \oplus PAM_{\psi,\gamma}(A; E).$$

Proof. By Theorem 3.5 we have $PAM_{\psi,\gamma}(A\wr B)=PAM_{\psi,\gamma}(B)\oplus PAM_{\psi,\gamma}(D,B)$. Let b_i, $i=1,2,\dots,k$, be distinct elements from B. Then for any a_i, $i=1,2,\dots,k$, the subsemigroup of D generated by $a_i(b_i)$, $i=1,2,\dots,k$, is abelian. Hence if $u=a_1(b_1)a_2(b_2)\cdots a_k(b_k)$, $v=\alpha_1(b_1)\alpha_2(b_2)\cdots\alpha_k(b_k)$ and $f\in PAM_{\psi,\gamma}(D,B)$, then

$$\|f(uv)-f(u)-f(v)\|=\left\|\sum_{i=1}^{k}[f(a_i\alpha_i(b_i))-f(a_i(b_i))-f(\alpha_i(b_i))]\right\|.$$

Let B be an infinite group, and let b_i for $i\in\mathbb{N}$ be distinct elements from B. Let $a,\alpha\in A$. Consider elements $u_k=a(b_1)a(b_2)\cdots a(b_k)$ and $v_k=\alpha(b_1)\alpha(b_2)\cdots\alpha(b_k)$. Then by Theorem 2.10, for any $k\in\mathbb{N}$, we have

$$\|f(uv)-f(u)-f(v)\|=\left\|\sum_{i=1}^{k}[f(a\alpha(b_i))-f(a(b_i))-f(\alpha(b_i))]\right\|.$$

Let $r=f(a\alpha(b_i))-f(a(b_i))-f(\alpha(b_i))$. Hence,

$$\begin{aligned}\|f(uv)-f(u)-f(v)\|&=\left\|\sum_{i=1}^{k}[f(a\alpha(b_i))-f(a(b_i))-f(\alpha(b_i))]\right\|\\&=\|k[f(a\alpha(b_i))-f(a(b_i))-f(\alpha(b_i))]\|\\&=k\cdot\|r\|.\end{aligned}$$

Furthermore, we have

$$\|f(uv)-f(u)-f(v)\|\le c\,[\psi(\gamma(u))+\psi(\gamma(v))].$$

Hence,

$$\begin{aligned}k\|r\|&\le c[\psi(\gamma(u_k))+\psi(\gamma(v_k))]\\&=c\,[\psi(\gamma(a(b_1)a(b_2)\cdots a(b_k)))+\psi(\gamma(\alpha(b_1)\alpha(b_2)\cdots\alpha(b_k)))]\\&\le c\left[\psi\left(\sum_{i=1}^{k}\gamma(a(b_i))\right)+\psi\left(\sum_{i=1}^{k}\gamma(\alpha(b_i)\right)\right]\\&\le c\,[\psi(k\,\gamma(a))+\psi(k\,\gamma(\alpha))]\\&\le c\,[\psi(k)\,\psi(\gamma(a))+\psi(k)\,\psi(\gamma(\alpha))].\end{aligned}$$

Hence,

$$\|r\|\le c\,\frac{\psi(k)}{k}\,\psi(\gamma(a))+c\,\frac{\psi(k)}{k}\,\psi(\gamma(\alpha)),\quad \forall k\in\mathbb{N}.$$

The latter is possible only if $r=0$. Thus if B is infinite and $f\in PAM_{\psi,\gamma}(D,B)$, then f is a homomorphism of D. Denote by ξ_b the restriction of f to $A(b)$. Let a be an arbitrary element from A. According to the action of B on D we have

$$f(a(b))=f(a(1)^b)=f(a(1)),$$

that is, $\xi_b(a(b))=\xi_1(a(1))$. Hence, there is an element ξ in $HOM(A;E)$ such that $\xi_b(a(b))=\xi(a)$ for any $a\in A$ and for any $b\in B$. Therefore, for any $u=a_1(b_1)a_2(b_2)\cdots a_k(b_k)$ the relation

$$f(a_1(b_1)a_2(b_2)\cdots a_k(b_k))=\sum_{i=i}^{k}\xi(a_i)$$

holds. Hence $PAM_{\psi,\gamma}(A\wr B;E) = PAM_{\psi,\gamma}(B;E)\oplus HOM(A;E)$.

Now let B be a finite group of order m. It is easy to verify that in this case, for any $f\in PAM_{\psi,\gamma}(D,B)$, there is a $\varphi\in PAM_{\psi,\gamma}(A)$ such that

$$f(a_1(b_1)a_2(b_2)\cdots a_m(b_m)) = \sum_{i=i}^{m}\varphi(a_i).$$

Hence if the group B is finite, then

$$PAM_{\psi,\gamma}(A\wr B;E) = PAM_{\psi,\gamma}(B;E)\oplus PAM_{\psi,\gamma}(A;E).$$

□

Definition 4.2. We say that the functional equation

$$f(xy) - f(x) - f(y) = 0 \tag{4.1}$$

is (ψ,γ)–*stable* on a semigroup S if for any f satisfying the functional inequality

$$\|f(xy) - f(x) - f(y)\| \le \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))],\quad x,y\in S,$$

for some $\theta > 0$ there is a solution T of the functional equation (4.1) such that the function $T(x) - f(x)$ belongs to $B_{\psi,\gamma}(S;E)$.

It is evident that (4.1) is stable if and only if $PAM_{\psi,\gamma}(S;E) = HOM(S;E)$.

Now from Theorem 2.10 we obtain the following corollary.

Corollary 4.3. *The functional equation (4.1) is stable for any Abelian semigroup.*

Consider the free group F on two generators α,β. Let f be Forti's function (see the introduction). Then the function $\widehat{f}$ is a pseudocharacter of F such that $\widehat{f}(\alpha\beta) = 1$. Let E be an arbitrary Banach space. Consider a mapping $\varphi : F\to E$ such that $\varphi(x) = \widehat{f}(x)\cdot e$, where e denotes some fixed element of E such that $\|e\| = 1$. Let us define a function $\gamma : F\to\mathbb{R}$ as a constant $\gamma(x) = c$ for some $c > 0$.

Now let us verify that the equation (4.1) is not (ψ,γ)–stable.

Indeed, it is easy to see that $\varphi(\alpha) = \varphi(\beta) = 0$. Suppose that there is an element $\xi\in HOM(F;E)$ such that $g = \varphi - \xi\in B(F;E)$; then $g(\alpha) = \xi(\alpha)$, $g(\beta) = \xi(\beta)$. From the relations $g(\alpha^n) = \xi(\alpha^n) = n\cdot\xi(\alpha)$, $g(\beta^n) = n\cdot\xi(\beta)$, we obtain $\xi(\alpha) = \xi(\beta) = 0$. Hence $\xi\equiv 0$ on F. This implies $g = \varphi$ and contradicts the assumption that g is bounded.

So, in general, equation (4.1) is not (ψ,γ)–stable on the class of all groups.

Corollary 4.4. *Let A be an arbitrary group. Then A can be embedded into a group G, and the equation*

$$f(xy) - f(x) - f(y) = 0$$

is (ψ,γ)–stable on G.

Proof. If equation (4.1) is (ψ,γ)–stable on A, there is nothing to prove. Now suppose that (4.1) is not (ψ,γ)–stable on A. Let $G = A\wr\mathbb{Z}$. Then by Theorem 4.1, we obtain $PAM_{\psi,\gamma}(G;E) = PAM_{\psi,\gamma}(\mathbb{Z};E)\oplus HOM(A;E)$. Now from Theorem 2.10, we have $PAM_{\psi,\gamma}(\mathbb{Z};E) = HOM(\mathbb{Z};E)$. Hence,

$$PAM_{\psi,\gamma}(G;E) = HOM(\mathbb{Z};E)\oplus HOM(A;E),$$

and the proof is complete. □

Definition 4.5. We shall say that an element x of a semigroup S is *periodic* if there are $n, m \in \mathbb{N}$ such that $n \neq m$ and $x^n = x^m$. We shall say that the semigroup is periodic if every element of S is periodic.

From Lemma 2.8, it follows that if $f \in PAM_{\psi,\gamma}(S;E)$ and x is a periodic element of S, then $f(x) = 0$. Indeed, for some $n \neq m$, $f(x^n) = f(x^m)$ implies $nf(x) = mf(x)$, which is $(n-m)f(x) = 0$, and thus $f(x) = 0$. So equation (4.1) is stable for any periodic semigroup.

Remark 4.6. If S is a semigroup with zero and $f \in PAM_{\psi,\gamma}(S,E)$, then $f \equiv 0$.

To see this, consider, for some $\theta > 0$,

$$\|f(xy) - f(x) - f(y)\| \leq \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in S.$$

It is clear that $f(0) = 0$. Now substituting 0 for y in the last inequality, we obtain $\|f(x)\| \leq \theta\,[\psi(\gamma(x)) + \psi(\gamma(0))]$ for all $x \in S$. The latter implies $\|f(x^n)\| \leq \theta\,[\psi(\gamma(x^n)) + \psi(\gamma(0))]$ for all $n \in \mathbb{N}$. Hence,

$$\|f(x)\| \leq \theta\,\frac{1}{n}\,[\psi(n\,\gamma(x)) + \psi(\gamma(0))], \quad \forall n \in \mathbb{N},$$

and

$$\|f(x)\| \leq \theta\,\left[\frac{\psi(n)}{n}\,\psi(\gamma(x)) + \frac{1}{n}\,\psi(\gamma(0))\right] \quad \forall n \in \mathbb{N}.$$

The last inequality yields $f(x) = 0$ for all $x \in S$.

The next corollary follows from Remark 4.6.

Corollary 4.7. *Any semigroup S can be embedded into a semigroup S_0 such that equation (4.1) is (ψ,γ)–stable on S_0.*

Proof. Let S_0 be a semigroup obtained by adjoining the zero to the semigroup S. From Remark 4.6, we have $PAM_{\psi,\gamma}(S_0,E) = \{0\}$. Hence equation (4.1) is (ψ,γ)–stable on S_0. □

Definition 4.8. We shall say that in a semigroup S a *left law of reduction* is fulfilled if any equality $xy = xz$ in S implies $y = z$. Similarly, we shall say that in a semigroup S a *right law of reduction* is fulfilled if any equality $yx = zx$ in S implies $y = z$.

Obviously, in a semigroup with zero, neither a left nor a right law of reduction is fulfilled.

Theorem 4.9. *Let S be a semigroup with left (or right) law of reduction. Then S can be embedded into a semigroup G with the left (or right respectively) law of reduction such that equation (4.1) is (ψ,γ)–stable on G.*

Proof. Let S be a semigroup with left (or right) law of reduction. It is not difficult to verify that $S \wr \mathbb{Z}$ is a semigroup with left (or right) law of reduction respectively. Now we apply the same arguments as in the proof of Corollary 4.4. □

Let G be an arbitrary group. For $a, b, c \in G$, we set $[a,b] = a^{-1}b^{-1}ab$ and $[a,b,c] = [[a,b],c]$.

Definition 4.10. We shall say that G is *metabelian* if for any $x, y, z \in G$, we have $[[x,y],z] = 1$.

It is clear that if $[x,y]=1$, then $[[x,y],z]=1$, and hence any abelian group is metabelian.

Our next goal is to prove a stability theorem for any metabelian group. Consider the group H with two generators a,b and the following defining relations:

$$[b,a]a=a[b,a], \quad b[b,a]=[b,a]b. \tag{4.2}$$

If we set $c=[b,a]$, we get the following representation of H in terms of generators and defining relations: $H=\langle a,b,c \| c=[b,a],\ [c,a]=[c,b]=1\rangle$.

It is well known that each element of H can be uniquely represented as $g=a^m b^n c^k$, where $m,n,k\in\mathbb{Z}$. The mapping

$$g=a^m b^n c^k \to \begin{bmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism between H and $UT(3,\mathbb{Z})$.

By induction on p, it is easy to verify for any $m,n,k,p\in\mathbb{Z}$ the following relation holds:

$$(a^n b^m)^p = a^{np} b^{mp} c^{\rho(p)nm}, \quad \text{where } \rho(p)=\sum_{i=1}^{p-1} i.$$

From this relation it follows that

$$(a^n b^m c^k)^p = a^{np} b^{mp} c^{\rho(p)nm+pk}. \tag{4.3}$$

Lemma 4.11. *If $f\in PAM_{\psi,\gamma}(H;E)$ and $f(a)=f(b)=0$, then $f\equiv 0$.*

Proof. Let us verify that $b^{-1}a^{-1}ba=bab^{-1}a^{-1}$. Indeed, $ba=abc$; hence $bab^{-1}a^{-1}=abcb^{-1}a^{-1}=abb^{-1}a^{-1}c=c$. Now from Theorems 2.10, 2.11 and Lemma 2.8 we obtain $f(c)=f(b^{-1}a^{-1}ba)=f(b^{-1}a^{-1})+f(ba)=f((ab)^{-1})+f(ba)=-f(ab)+f(ba)=-f(ba)+f(ba)=0$.

The element c belongs to the center of H. Now for $z=a^m b^n c^k$, we have $f(z)=f(a^m b^n c^k)=f(a^m b^n)+f(c^k)=f(a^m b^n)$.

Therefore,

$$\|f(z)-f(a^m)-f(b^n)\| \le \theta\,[\psi(\gamma(a^m))+\psi(\gamma(b^n))],$$
$$\|f(z)\| \le \theta\,[\psi(m\,\gamma(a))+\psi(n\,\gamma(b))],$$

and now from the relation (4.3), we obtain

$$\|f(z^p)\| \le \theta\,[\psi(m\,p\,\gamma(a))+\psi(n\,p\,\gamma(b))],$$
$$\|f(z)\| \le \theta\,\tfrac{\psi(p)}{p}\,[\psi(m\,\gamma(a))+\psi(n\,\gamma(b))].$$

Now from the relation $\frac{\psi(p)}{p}\to 0$ as $p\to\infty$, we get $f(z)=0$. □

Lemma 4.12. $PAM_{\psi,\gamma}(H;E)=HOM(H;E)$.

Proof. It is well known that the group H is a free metabelian group and the elements a,b are free generators of H. This means that every assignment $a\to x$ and $b\to y$, where $x,y\in E$, can be extended to a homomorphism of H into E. Now let $f\in PAM_{\psi,\gamma}(H;E)$; then there is an element $\varphi\in HOM(H;E)$ such that $\varphi(a)=f(a)$, $\varphi(b)=f(b)$. Since $g=f-\varphi\in PAM_{\psi,\gamma}(H;E)$ and $g(a)=g(b)=0$, by the previous lemma we get $g\equiv 0$. Therefore $f\equiv\varphi$. □

Theorem 4.13. *The equation (4.1) is stable on any metabelian group.*

Proof. Let G be a metabelian group and $f \in PAM_{\psi,\gamma}(G;E)$. Let $x, y \in G$; then there is a homomorphism τ of H into G such that $\tau(a) = x$ and $\tau(b) = y$. It is clear that the function γ^* on H defined by the rule $\gamma^*(g) = \gamma(\tau(g))$ satisfies the condition $\gamma^*(g_1 g_2) \le \gamma^*(g_1) + \gamma^*(g_2)$. Obviously, the function $f^*(g) = f(\tau(g))$ belongs to $PAM_{\psi,\gamma^*}(H;E)$. Now if $f(xy) \ne f(x)+f(y)$, then $f^*(ab) \ne f^*(a)+f^*(b)$, and we arrive at a contradiction with the previous lemma. Thus $f \in HOM(H;E)$. □

Acknowledgment

The authors would like to thank the referee for suggestions that improved the presentation of this paper.

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Tver State Agricultural Academy, Tver Sakharovo, Russia
Current address: Tereshkova Sq. 47/27–43, Tver 170037, Russia
E-mail address: valeriy.fayziev@tversu.ru

Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece
E-mail address: trassias@math.ntua.gr

Department of Mathematics, University of Louisville, Louisville, Kentucky 40292
E-mail address: sahoo@louisville.edu